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SPHERICAL REPRESENTATION OF CONJUGATE SYSTEMS AND ASYMPTOTIC LINES.*

BY W. C. GRAUSTEIN

1. **Introduction.** A system of curves on the Gauss sphere represents, as is well known, a conjugate system of curves on each of infinitely many surfaces. It is shown in this paper that, if the ratio of the radii of normal curvature in the conjugate directions is prescribed subject to a certain condition, *one* of the required surfaces is determined to within its homothetics (§ 2).

Since the condition in question can be written in a form involving only the differences of the point and of the plane invariants of the conjugate system and an expression whose vanishing is the condition that the conjugate system be isothermal-conjugate, the general result is particularly adaptable to the important special cases of conjugate systems which have equal invariants of either kind or are isothermal-conjugate (§§ 3, 4). It is also readily applied to translation surfaces (§ 5).

In the case of lines of curvature the condition can be put into a second equally striking form containing, besides the differences of the point and of the plane invariants, merely an expression whose vanishing is the condition that the ratio of the principal radii of curvature is of the form $U(u)/V(v)$, the lines of curvature being parametric. Inasmuch as the lines of curvature, or their spherical representation, form an isothermal (orthogonal) system, according as the point invariants, or the plane invariants, are equal, the general result again leads immediately to important special theorems (§ 6).

One family of a system of curves on the Gauss sphere represents one family of asymptotic lines on each of infinitely many surfaces having the prescribed spherical representation. In § 7 it is proved that, if the second family of asymptotic lines is prescribed subject to a certain condition, *one* of the surfaces is determined to within its homothetics. Thus a generalization of Dini's Theorem† is obtained. Applications of it are made to ruled and minimal surfaces.

The content of the paper is closely related to the theory of parallel maps and can, in fact, be used as the basis of a development of this theory. For, a parallel map can be considered as a one-to-one point correspondence between

* Presented to the American Mathematical Society, February 25, 1922.

† Cf., e. g., Eisenhart, *Differential Geometry*, p. 192.

two surfaces such that their spherical representations are identical; furthermore, in the theory of a general parallel map the corresponding conjugate systems and the quotient of the ratios of the radii of normal curvature in the corresponding conjugate directions play determinative roles.*

It is assumed throughout that the surfaces considered are real, non-developable, and analytic. The families of curves constituting a conjugate system can be taken as real or, provided the minimal curves of a minimal surface are excluded,† as conjugate-imaginary.

2. Conjugate system arbitrary. If a surface S , $x = x(u, v)$:

$$x_1 = x_1(u, v), \quad x_2 = x_2(u, v), \quad x_3 = x_3(u, v),$$

is referred to a conjugate system as the parametric curves, the Codazzi equations can be written in the form

$$(1) \quad \frac{\partial \log e}{\partial v} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{g}{e}, \quad \frac{\partial \log g}{\partial u} = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{e}{g},$$

where $e, f (= 0), g$ are the differential coefficients of the second order of S and the Christoffel symbols pertain to the linear element,

$$\mathfrak{E} du^2 + 2\mathfrak{F} du dv + \mathfrak{G} dv^2,$$

of the spherical representation of S . From equations (1) we obtain the relation

$$(2) \quad \frac{\partial^2 \log e/g}{\partial u \partial v} = \frac{\partial}{\partial u} \left[\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{g}{e} \right] - \frac{\partial}{\partial v} \left[\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{e}{g} \right].$$

Since

$$(3) \quad E = \frac{e^2 \mathfrak{G}}{\mathfrak{G}^2}, \quad F = -\frac{eg \mathfrak{F}}{\mathfrak{G}^2}, \quad G = \frac{g^2 \mathfrak{E}}{\mathfrak{G}^2},$$

* Cf. Author, "Parallel maps of surfaces", Trans. Amer. Math. Soc., vol. 23 (1922), pp. 298-332, in particular §§ 19, 20, where the results of the present paper are used in developing conditions for the existence of a parallel map when the spherical representation, together with certain other definitive elements, is given. It is to be noted that the theory of the present § 7, concerning asymptotic lines, comes into play in the case of a parallel map for which the corresponding conjugate systems have become single families of asymptotic lines.

† The ratio of the radii of normal curvature in a pair of minimal conjugate directions is undefined.

where $\mathfrak{G}^2 = \mathfrak{E}\mathfrak{G} - \mathfrak{F}^2$, it follows that

$$(4) \quad \frac{e}{g} = \frac{\mathfrak{E}}{\mathfrak{G}} \frac{R_1}{R_2},$$

where R_1 and R_2 are the radii of normal curvature of S in the given conjugate directions. Hence the ratio R_1/R_2 satisfies the equation:

$$(5) \quad \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{E}}{\mathfrak{G}} \frac{R_1}{R_2} = \frac{\partial}{\partial u} \left[\frac{\{12\}'}{\{1\}} - \frac{\mathfrak{G}}{\mathfrak{E}} \frac{\{11\}'}{\{2\}} \frac{R_2}{R_1} \right] - \frac{\partial}{\partial v} \left[\frac{\{12\}'}{\{2\}} - \frac{\mathfrak{E}}{\mathfrak{G}} \frac{\{22\}'}{\{1\}} \frac{R_1}{R_2} \right].$$

Conversely, if there is given on the Gauss sphere a system of curves $\mathfrak{E}$ and a point function R_1/R_2 such that, when the curves $\mathfrak{E}$ are parametric*, $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}$ and R_1/R_2 satisfy (5), equations (1) and (4), in e and g , are compatible; e and g can be found by a quadrature and are unique except for the same multiplicative constant, k . Consequently, the surface S is determined to within its homothetics.

To find the parametric equations, $x = x(u, v)$, of S , it is necessary first to find those of the sphere, $\zeta = \zeta(u, v)$, by solving a Riccati equation. Then x_1, x_2, x_3 can be obtained by quadratures from the equations

$$\frac{\partial x}{\partial u} = \frac{e}{\mathfrak{G}^2} \left(-\mathfrak{G} \frac{\partial \zeta}{\partial u} + \mathfrak{F} \frac{\partial \zeta}{\partial v} \right), \quad \frac{\partial x}{\partial v} = \frac{g}{\mathfrak{G}^2} \left(\mathfrak{F} \frac{\partial \zeta}{\partial u} - \mathfrak{E} \frac{\partial \zeta}{\partial v} \right),$$

and are determined except for the multiplier k and additive constants.

THEOREM 1. *A system of curves on the sphere represents a conjugate system on a surface for which the ratio R_1/R_2 of the radii of normal curvature in the conjugate directions is prescribed if and only if $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, R_1/R_2$ satisfy (5). The surface is then determined to within its homothetics and its point coordinates can be found by quadratures, when those of the sphere are known.*

Since between the Christoffel symbols for S and those for the spherical representation the relations

$$(6) \quad \frac{\{12\}}{\{1\}} = -\frac{g}{e} \frac{\{11\}'}{\{2\}}, \quad \frac{\{12\}}{\{2\}} = -\frac{e}{g} \frac{\{22\}'}{\{1\}}$$

subsist, (2) can be written in the form

$$(7) \quad \frac{\partial^2 \log e/g}{\partial u \partial v} = (h - k) + (h' - k'),$$

* Explicit mention of this condition, which we shall always assume fulfilled, will henceforth be suppressed.

where

$$(8) \quad h - k = \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}, \quad h' - k' = \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}',$$

and h, k are the point invariants and h', k' the plane invariants of the given conjugate system.

3. Isothermal-conjugate systems. If the parametric curves on the surface S form an isothermal-conjugate system, e/g is of the form $U(u)/V(v)$ and (5) splits into the two equations:

$$(9) \quad \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{E}}{\mathfrak{G}} \frac{R_1}{R_2} = 0,$$

$$\frac{\partial}{\partial u} \left[\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - \frac{\mathfrak{G}}{\mathfrak{E}} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{R_2}{R_1} \right] = \frac{\partial}{\partial v} \left[\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' - \frac{\mathfrak{E}}{\mathfrak{G}} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{R_1}{R_2} \right].$$

THEOREM 2. *A necessary and sufficient condition that there exist a surface S having an isothermal-conjugate system which is represented by a given system of curves on the sphere is that equations (9), in R_1/R_2 , be compatible. Then to each solution, R_1/R_2 , there corresponds a surface S which is unique to within its homothetics and the ratio of whose radii of normal curvature in the isothermal-conjugate directions is R_1/R_2 .*

If the given system of curves on the sphere is to represent an isothermal-conjugate system and the parameters u, v are to be isothermal-conjugate as well, that is, if e/g is to have the value $+1$ or -1 , according as S is to be of positive or negative curvature, equations (9) reduce to the single, well known condition:*

$$\frac{\partial}{\partial u} \left[\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \right] = \frac{\partial}{\partial v} \left[\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \right].$$

If this condition is satisfied, S is unique to within its homothetics and $R_1/R_2 = \pm \mathfrak{G}/\mathfrak{E}$.

From (7) we conclude the following:

THEOREM 3. *A necessary and sufficient condition that a conjugate system be isothermal-conjugate is that the difference of its point invariants equal the difference of its plane invariants taken in the opposite order.*

4. Conjugate systems with equal invariants. By using (7), in conjunction with (4) and (6), we obtain from Theorem 1 the following results.

* Cf., e. g., Eisenhart, *Differential Geometry*, p. 202.

THEOREM 4. *A necessary and sufficient condition that there exist a surface S having a conjugate system with equal point invariants which is represented by a given system of curves on the sphere is that the equations*

$$(10) \quad \frac{\partial}{\partial u} \left[\frac{\mathfrak{G}}{\mathfrak{E}} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{R_2}{R_1} \right] = \frac{\partial}{\partial v} \left[\frac{\mathfrak{E}}{\mathfrak{G}} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{R_1}{R_2} \right], \quad \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{E}}{\mathfrak{G}} \frac{R_1}{R_2} = h' - k',$$

in R_1/R_2 , be compatible. Then to each solution, R_1/R_2 , there corresponds a surface S which is unique to within its homothetics and has R_1/R_2 as the ratio of its radii of normal curvature in the conjugate directions.

The given curves on the sphere represent a conjugate system with equal plane invariants on some surface S if and only if $h' = k'$; then each solution of

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{E}}{\mathfrak{G}} \frac{R_1}{R_2} = \frac{\partial}{\partial v} \left[\frac{\mathfrak{E}}{\mathfrak{G}} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{R_1}{R_2} \right] - \frac{\partial}{\partial u} \left[\frac{\mathfrak{G}}{\mathfrak{E}} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{R_2}{R_1} \right]$$

leads to a surface S with the desired properties

In order that the conjugate system have both equal point invariants and equal plane invariants, it is necessary and sufficient that $h' = k'$ and that the first equations in (9) and (10) be compatible in R_1/R_2 . In this connection we note, from (7), the following known result:

THEOREM 5. *If a conjugate system has two of the three properties, (a) of having equal point invariants, (b) of having equal plane invariants, (c) of being isothermal-conjugate, it has also the third.*

5. Translation surfaces. A surface is a translation surface referred to its generators as the parametric curves if and only if

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = 0, \quad \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = 0.$$

Accordingly, we obtain, from (6) and (5), the following result

THEOREM 6. *A necessary and sufficient condition that a system of curves on the sphere represent the generators of a translation surface is that*

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = 0, \quad \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' = 0.$$

Then each solution, R_1/R_2 , of the second equation of (10) determines such a surface to within its homothetics and the ratio of the radii of normal curvature in the directions of the generators is R_1/R_2 .

6. Lines of curvature. If the given conjugate system on the surface S consists of the lines of curvature, we obtain, from (3) and (4), the equations

$$(11) \quad \frac{e}{g} = \frac{\mathfrak{E}}{\mathfrak{G}} \frac{r_1}{r_2}, \quad \frac{E}{G} = \frac{\mathfrak{E}}{\mathfrak{G}} \left(\frac{r_1}{r_2} \right)^2,$$

where r_1 and r_2 are the principal radii of curvature. Since now

$$(12) \quad h - k = \frac{1}{2} \frac{\partial^2 \log E/G}{\partial u \partial v}, \quad h' - k' = \frac{1}{2} \frac{\partial^2 \log \mathfrak{E}/\mathfrak{G}}{\partial u \partial v},$$

we find, from the second equation of (11), that

$$(13) \quad \frac{\partial^2 \log r_1/r_2}{\partial u \partial v} = (h - k) - (h' - k').$$

But relations (6) become

$$(14) \quad \frac{\partial \log E}{\partial v} = \frac{r_2}{r_1} \frac{\partial \log \mathfrak{E}}{\partial v}, \quad \frac{\partial \log G}{\partial u} = \frac{r_1}{r_2} \frac{\partial \log \mathfrak{G}}{\partial u},$$

and so (13) can be written in the form

$$(15) \quad 2 \frac{\partial^2 \log r_1/r_2}{\partial u \partial v} = \frac{\partial}{\partial u} \left[\left(\frac{r_2}{r_1} - 1 \right) \frac{\partial \log \mathfrak{E}}{\partial v} \right] - \frac{\partial}{\partial v} \left[\left(\frac{r_1}{r_2} - 1 \right) \frac{\partial \log \mathfrak{G}}{\partial u} \right],$$

which is the equation to which (5) reduces in this case.

THEOREM 7. *An orthogonal system of curves on the sphere represents the lines of curvature on a surface for which the ratio r_1/r_2 of the principal radii of curvature is prescribed, if and only if $\mathfrak{E}$, $\mathfrak{G}$, r_1/r_2 satisfy (15). The surface is then determined to within its homothetics.*

Consider now the four properties: (a) the ratio of the principal radii of curvature of the form $U(u)/V(v)$, the lines of curvature being para-

metric; (b) the lines of curvature an isothermal-orthogonal system; (c) the lines of curvature an isothermal-conjugate system; (d) the spherical representation of the lines of curvature an isothermal system. From (11) follows immediately the theorem:

THEOREM 8. *If a surface has any two of the properties enumerated, it has the other two also.**

If $r_1/r_2 = -1$, we find, from (15), that $\mathfrak{E}/\mathfrak{G}$ is a function of the form $U(u)/V(v)$.

THEOREM 9. *The lines of curvature of a minimal surface are isothermal, both as an orthogonal and as a conjugate system, and are represented on the sphere by an isothermal system. Conversely, an isothermal (orthogonal) system on the sphere represents the lines of curvature on a minimal surface unique to within its homothetics.*

From (15), in conjunction with (11), we obtain the following results.

THEOREM 10. *An orthogonal system of curves on the sphere represents the lines of curvature on a surface of Bour for which the ratio r_1/r_2 is prescribed, if and only if $\frac{\mathfrak{E}}{\mathfrak{G}} \left(\frac{r_1}{r_2}\right)^2$ is of the form $U(u)/V(v)$ and $\mathfrak{E}, \mathfrak{G}, r_1/r_2$ satisfy*

$$(16) \quad \frac{\partial}{\partial u} \left(\frac{r_2}{r_1} \frac{\partial \log \mathfrak{E}}{\partial v} \right) = \frac{\partial}{\partial v} \left(\frac{r_1}{r_2} \frac{\partial \log \mathfrak{G}}{\partial u} \right).$$

If, however, the lines of curvature are to form an isothermal-conjugate system, it is $\frac{\mathfrak{E}}{\mathfrak{G}} \frac{r_1}{r_2}$ which is to be of the form $U(u)/V(v)$; in this case $\mathfrak{E}, \mathfrak{G}, r_1/r_2$ must satisfy

$$\frac{\partial}{\partial u} \left[\left(\frac{r_2}{r_1} + 1 \right) \frac{\partial \log \mathfrak{E}}{\partial v} \right] = \frac{\partial}{\partial v} \left[\left(\frac{r_1}{r_2} + 1 \right) \frac{\partial \log \mathfrak{G}}{\partial u} \right].$$

Finally, if the lines of curvature are to be isothermal, both as an orthogonal and a conjugate system, both $\mathfrak{E}/\mathfrak{G}$ and r_1/r_2 must be of the form $U(u)/V(v)$ and $\mathfrak{E}, \mathfrak{G}, r_1/r_2$ must satisfy (16).

THEOREM 11. *A necessary and sufficient condition that an isothermal (orthogonal) system of curves on the sphere represent an isothermal-orthogonal or*

* Eisenhart has shown that if a surface has any two of the last three properties, it has the third also, "Isothermal-conjugate lines on surfaces", Amer. Journ. Math., vol. 25 (1903), pp. 213-248, in particular, p. 228. From Eisenhart's result can be deduced the first part of Theorem 9, since it is well known that the lines of curvature of a minimal surface and their spherical representation form isothermal-orthogonal systems.

isothermal-conjugate system of lines of curvature on a surface for which the ratio r_1/r_2 is prescribed is that r_1/r_2 be of the form $U(u)/V(v)$ and $\mathfrak{E}, \mathfrak{G}, r_1/r_2$ satisfy (16).

Equation (13), which is the equivalent in this case of the general equation (7), embodies the following result.

THEOREM 12. *When the lines of curvature on a surface are parametric, the ratio of the principal radii of curvature is of the form $U(u)/V(v)$ if and only if the difference of the point invariants of the lines of curvature equals the difference of their plane invariants.*

7. Asymptotic lines. If a surface S of negative curvature is referred to one family of asymptotic lines as the u -curves and to any other family of real curves as the v -curves, the Codazzi equations become

$$\frac{\partial f}{\partial u} = \left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' \right] f + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' g,$$

$$\frac{\partial f}{\partial v} - \frac{\partial g}{\partial u} = \left[\begin{Bmatrix} 22 \\ 2 \end{Bmatrix}' - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' \right] f - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' g.$$

We divide each equation by f , replace the second by the equation obtained by adding the first multiplied by g/f to the second, and simplify by means of the identities,

$$\frac{\partial \log \mathfrak{F}}{\partial v} = \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}' + \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}', \quad \frac{\partial \log \mathfrak{F}}{\partial u} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}'.$$

The result is

$$-\frac{\partial \log f/\mathfrak{F}}{\partial u} = 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{g}{f},$$

(17)

$$\frac{\partial \log f/\mathfrak{F}}{\partial v} = \frac{\partial}{\partial u} \frac{g}{f} - 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' + \left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' - 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' \right] \frac{g}{f} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \left(\frac{g}{f} \right)^2.$$

Hence

$$(18) \quad \frac{\partial^2}{\partial u^2} \frac{g}{f} - \frac{\partial}{\partial u} \left\{ 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' - \left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' - 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' \right] \frac{g}{f} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \left(\frac{g}{f} \right)^2 \right\} \\ + \frac{\partial}{\partial v} \left[2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{g}{f} \right] = 0.$$

Since $e = 0$, the second family of asymptotic lines is defined by the equation

$$du + M dv = 0,$$

where

$$(19) \quad M = \frac{g}{2f}.$$

Equation (18), written in terms of M , is

$$(20) \quad \frac{\partial^2 M}{\partial u^2} - \frac{\partial}{\partial u} \left\{ \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - \left[\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' - 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \right] M - 2 \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' M^2 \right\} \\ + \frac{\partial}{\partial v} \left[\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' M \right] = 0.$$

Conversely, if there is given on the Gauss sphere a system of curves, parametric, and a point function M , such that $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, M$ satisfy (20), equations (17) in f , where g/f has been replaced, by $2M$, are integrable by a quadrature and determine f except for a multiplicative constant, k ; $g = 2fM$ is then unique except for the same multiplier, k . Consequently, the surface S is determined to within its homothetics. Its point coördinates, when those of the sphere are known, can be found by quadratures from the equations

$$\frac{\partial x}{\partial u} = \frac{f}{\mathfrak{E}^2} \left(\mathfrak{F} \frac{\partial \xi}{\partial u} - \mathfrak{E} \frac{\partial \xi}{\partial v} \right), \quad \frac{\partial x}{\partial v} = \frac{1}{\mathfrak{E}^2} \left[(\mathfrak{F}g - \mathfrak{G}f) \frac{\partial \xi}{\partial u} + (\mathfrak{F}f - \mathfrak{E}g) \frac{\partial \xi}{\partial v} \right],$$

and are determined except for the multiplier k and additive constants.

THEOREM 13. *The u -curves of a real parametric system on the sphere represent one family of asymptotic lines on a surface for which the differential equation $du + M dv = 0$ of the second family of asymptotic lines is prescribed if and only if $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, M$ satisfy (20). The surface is then determined to within its homothetics and its point coördinates can be found by quadratures, when those of the sphere are known.*

If $M = 0$, that is, if the given system on the sphere is to represent both families of asymptotic lines, condition (20) reduces to

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'.$$

We thus obtain Dini's Theorem* as a special case.

* Cf. Introduction.

Since, when $e = 0$,

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\},$$

and the vanishing of the latter symbol is a necessary and sufficient condition that the u -curves be straight, we conclude the following:

THEOREM 14. *The u -curves of a real parametric system on the sphere represent the rulings of a ruled surface for which the differential equation $du + M dv = 0$ of the second family of asymptotic lines is prescribed, if and only if $\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = 0$ and $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, M$ satisfy the equation*

$$(21) \quad \frac{\partial^2 M}{\partial u^2} - \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - \left[\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' - 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \right] M + \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' = 0.$$

If the given surface is minimal, $M = \mathfrak{F}/\mathfrak{E}$. Consequently, the u -curves of a system on the sphere represent one family of asymptotic lines on a minimal surface if and only if, when M in (20) is replaced by $\mathfrak{F}/\mathfrak{E}$, the equation is satisfied.

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